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Plane torsion waves in quadratic gravitational theories

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Abstract

The definition of the Riemann–Cartan space of the plane wave type is given. The condition under which the torsion plane waves exist is found. It is expressed in the form of the restriction imposed on the coupling constants of the 10-parametric quadratic gravitational Lagrangian. In the mathematical appendix the formula for commutator of the variation operator and Hodge operator is proved. This formula is applied for the variational procedure when the gravitational field equations are obtained in terms of the exterior differential forms.

1. Introduction

Recently the more attention is payed to the investigation of exact solutions of field equations in the spaces with the geometrical structure which is more complicated that the Riemann structure of General Relativity. The problem of the wavelike solutions is on the special place here since this problem is closely connected with the experimental investigation of gravitational waves. In [1] the gravitational waves of a metric and torsion are considered in the theory with the Lagrangian which represents the sum of the linear Einstein–Cartan Lagrangian, one of the six possible quadratic in curvature terms and all the possible terms quadratic in torsion. In [2] the torsion waves on the flat space background are described. The Lagrangian considered there is quadratic in curvature. In [3] the authors investigate the plane waves in the framework of the theory based on the Lagrangian quadratic in curvature and torsion without linear term. Ref. [4] is devoted to the investigation of the waves of the torsion 2-form of the algebraic special N-type.

In this article we consider the gravitational theory based on the general quadratic Lagrangian in the Riemann–Cartan space-time U_4. The main purpose is to investigate the problem of existance of the plane torsion waves in U_4. We also want to clearify the role of each irreducible part of the torsion propagating in the form of the plane wave. Preceding version of this work can be found in Ref. [5].

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The modern Poincaré gauge gravitational theory essentially uses the non-linear in curvature and torsion Lagrangians [6]–[22] (see also Refs. [24]–[26] and references therein). The use of the quadratic Lagrangians in the theory of gravitation is also stimulated by the attempt to construct the renormalized theory of gravitation in the Riemann–Cartan space-time [9], [25]. The most of the quadratic gravitational theories in the Riemann–Cartan space-time can be described as the particular cases of the 10-parametric Lagrangian, described in Ref. [8]. This Lagrangian is constructed as the sum of the Einstein–Cartan linear Lagrangian and all the terms quadratic in the irreducible pieces of curvature and torsion.

2. Field equations for the general quadratic Lagrangian

The Riemann–Cartan space-time U_4 is the 4-dimensional oriented differential manifold $\mathcal{M}$ with a metric g_{ab} $(a, b = 0, 1, 2, 3)$ (the metric has the Index 1), the volume 4-form η, the linear metric-compatible connection 1-form $\Gamma^a{}_b$, the curvature 2-form $\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd} \theta^c \wedge \theta^d$ and the torsion 2-form $\mathcal{T}^a = \frac{1}{2} T^a{}_{bc} \theta^b \wedge \theta^c$. Here θ^a $(a = 0, 1, 2, 3,)$ represent the cobasis of 1-forms in U_4 ($\wedge$ is the operation of the exterior product). We use the local anholonomic vector basis $\bar{e}_b$ with $\bar{e}_b \rfloor \theta^a = \delta^a_b$, where $\rfloor$ is the operation of the interior product, and $g_{ab} = g(\bar{e}_a, \bar{e}_b) = const.$

In U_4 a connection is compatible with a metric in the sence that the exterior covariant differential $D = d + \Gamma \wedge \ldots$ (d is the operator of the exterior differential) of the metric tensor is equal to zero,

$$Dg_{ab} = dg_{ab} - \Gamma^c{}_a g_{cb} - \Gamma^c{}_b g_{ac} = -2\Gamma_{(ab)} = 0 . \tag{2.1}$$

The condition (2.1) represents the constraint imposed on the connection 1-form $\Gamma^a{}_b$. This constraint can be resolved explicitly by using the connection 1-form which satisfies to the condition $\Gamma_{ab} = -\Gamma_{ba}$.

It is convinient to use the fields of 3-forms η_a, 2-forms η_{ab}, 1-forms η_{abc} and 0-forms η_{abcd}, determined as follows [27],

$$\eta_a = \bar{e}_a \rfloor \eta = *\theta_a , \qquad \eta_{ab} = \bar{e}_b \rfloor \eta_a = *(\theta_a \wedge \theta_b) , \tag{2.2}$$

$$\eta_{abc} = \bar{e}_c \rfloor \eta_{ab} = *(\theta_a \wedge \theta_b \wedge \theta_c) , \qquad \eta_{abcd} = \bar{e}_d \rfloor \eta_{abc} = *(\theta_a \wedge \theta_b \wedge \theta_c \wedge \theta_d) , \tag{2.3}$$

$$\theta^a \wedge \eta_b = \delta^a_b \eta , \qquad \theta^a \wedge \eta_{bc} = -2\delta^a_{[b} \eta_{c]} , \tag{2.4}$$

$$\theta^d \wedge \eta_{abc} = 3\delta^d_{[a} \eta_{bc]} , \qquad \theta^f \wedge \eta_{abcd} = -4\delta^f_{[a} \eta_{bcd]} , \tag{2.5}$$

where $*$ is the Hodge operator. In U_4 the following formula is valid [27],

$$D\eta_{ab} = \mathcal{T}^c \wedge \eta_{abc} . \tag{2.6}$$

Let us consider in U_4 the general 10-parametric Lagrangian which is constructed from all the terms quadratic in the irreducible pieces of curvature and torsion with the usual linear Einstein–Cartan term added. In terms of exterior differential forms it reads [29],

$$\mathcal{L} = f_0 \mathcal{R}^a{}_b \wedge \eta_a{}^b + \sum_{i=1}^{6} \lambda_i \overset{(i)}{\mathcal{R}}{}^a{}_b \wedge * \overset{(i)}{\mathcal{R}}{}_a{}^b + \sum_{i=1}^{3} \chi_i \overset{(i)}{\mathcal{T}}{}^a \wedge * \overset{(i)}{\mathcal{T}}{}_a . \tag{2.7}$$

Here $f_0 = 1/(2\kappa)$ $(\kappa = 8\pi G/c^4)$, and λ_i, χ_i are the coupling constants. The index (i) runs over all irreducible (with respect to Lorentz group) components of the curvature 2-form and the torsion 2-form, respectively. Refs. [1], [3] and some others deal with the special cases of the Lagrangian (2.7).

The torsion 2-form can be decomposed into traceless 2-form, trace 2-form and pseudotrace 2-form as follows (see [26]),

$$\mathcal{T}^a = \overset{(1)}{\mathcal{T}}{}^a + \overset{(2)}{\mathcal{T}}{}^a + \overset{(3)}{\mathcal{T}}{}^a .\tag{2.8}$$

Here the trace 2-form and the pseudotrace 2-form of the pseudo-Riemannian 4-manifold are determined by the expressions, respectively,

$$\overset{(2)}{\mathcal{T}}{}^a = \frac{1}{3}\theta^a \wedge (\bar{e}_b \rfloor \mathcal{T}^b) , \qquad \overset{(3)}{\mathcal{T}}{}^a = \frac{1}{3} * (\theta^a \wedge *(\mathcal{T}^b \wedge \theta_b)) .\tag{2.9}$$

The traceless and trace parts of torsion satisfy to the conditions,

$$\overset{(1)}{\mathcal{T}}{}^a \wedge \theta_a = 0 , \qquad \overset{(2)}{\mathcal{T}}{}^a \wedge \theta_a = 0 .\tag{2.10}$$

On the decomposition of the curvature 2-form into the irreducible pieces see Ref. [26].

Using the expressions for the irreducible parts of curvature and torsion 2-forms the Lagrangian (2.7) can be transformed into the form which is more convinient for the variation procedure,

$$\begin{aligned}
\mathcal{L} =\ & f_0 \mathcal{R}^a{}_b \wedge \eta_a{}^b + \tau_1 \mathcal{R}^a{}_b \wedge *\mathcal{R}^b{}_a \\
& +\tau_2 (\mathcal{R}^{ab} \wedge \theta_a) \wedge *(\mathcal{R}^c{}_b \wedge \theta_c) + \tau_3 (\mathcal{R}^{ab} \wedge \theta_c) \wedge *(\mathcal{R}^c{}_b \wedge \theta_a) \\
& +\tau_4 (\mathcal{R}^a{}_b \wedge \theta_a \wedge \theta^b) \wedge *(\mathcal{R}^c{}_d \wedge \theta_c \wedge \theta^d) \\
& +\tau_5 (\mathcal{R}^a{}_b \wedge \theta_a \wedge \theta^d) \wedge *(\mathcal{R}^c{}_d \wedge \theta_c \wedge \theta^b) \\
& +\tau_6 (\mathcal{R}^a{}_b \wedge \theta_c \wedge \theta^d) \wedge *(\mathcal{R}^c{}_d \wedge \theta_a \wedge \theta^b) + \varrho_1 \mathcal{T}^a \wedge *\mathcal{T}_a \\
& +\varrho_2 (\mathcal{T}^a \wedge \theta_a) \wedge *(\mathcal{T}^b \wedge \theta_b) + \varrho_3 (\mathcal{T}^a \wedge \theta_b) \wedge *(\mathcal{T}^b \wedge \theta_a) .
\end{aligned}\tag{2.11}$$

Here $\tau_1, ..., \tau_6, \varrho_1, ..., \varrho_3$ are the coupling constants. They are related to the coupling constants of the Lagrangian (2.7) as follows:

$$\tau_1 = \frac{1}{2}(3\lambda_4 - 5\lambda_1) , \qquad \tau_2 = \frac{1}{2}(\lambda_2 + \lambda_4 + \lambda_5 - 3\lambda_1) , \qquad \tau_3 = \frac{1}{2}(\lambda_4 - \lambda_1) ,$$

$$\tau_4 = \frac{1}{2}(-\lambda_2 + \lambda_3) , \quad \tau_5 = \frac{1}{2}(\lambda_2 + \lambda_4 - \lambda_5 - \lambda_1) , \quad \tau_6 = \frac{1}{12}(3\lambda_4 - 2\lambda_1 + \lambda_6) ,$$

$$\varrho_1 = \frac{1}{2}(2\chi_1 + \chi_2) , \qquad \varrho_2 = \frac{1}{3}(-\chi_1 + \chi_3) , \qquad \varrho_3 = \frac{1}{3}(\chi_1 - \chi_2) ,$$

One can easily express the parameters of (2.7) through the parameters of the Lagrangian (2.11). We need only one of them, namely,

$$\chi_1 = \varrho_1 + \varrho_3 .\tag{2.12}$$

Beeing written in the component form, the Lagrangian (2.11) coinsides with the well-known Lagrangian [8], $\mathcal{L} = L\eta$, where

$$\begin{aligned}
L =\ & f_0 R + R^{abcd}(f_1 R_{abcd} + f_2 R_{acbd} + f_3 R_{cdab}) \\
& + R^{ab}(f_4 R_{ab} + f_5 R_{ba}) + f_6 R^2 + T^{abc}(a_1 T_{abc} + a_2 T_{cba}) + a_3 T_a T^a .
\end{aligned}\tag{2.13}$$

Here we use the following notations:

$$R_{ab} = R^c{}_{acb} , \qquad R = R^c{}_c , \qquad T_a = T^c{}_{ac} .\tag{2.14}$$

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The constants in the Lagrangian (2.13) are related to the parameters of the Lagrangian (2.11) as follows,

$$f_1 = \frac{1}{2}(-\tau_1 + \tau_2 - \tau_3 + 2\tau_4 + \tau_5 + 2\tau_6) \,, \qquad f_2 = \tau_4 \,,$$

$$f_3 = -2\tau_2 - 4\tau_4 - \tau_5 \,, \quad f_4 = -\tau_3 - \tau_5 - 4\tau_6 \,, \quad f_5 = \tau_5 \,, \quad f_6 = \tau_6 \,,$$

$$a_1 = \frac{1}{2}(\varrho_1 + \varrho_3 + \varrho_3) \,, \qquad a_2 = -\varrho_2 \,, \qquad a_3 = -\varrho_3 \,.$$

The vacuum field equations in the Riemann–Cartan space-time can be obtained by the variational procedure of the first order. Let us vary the Lagrangian (2.11) with respect to the basis 1-form θ^a and to the connection 1-form Γ^a_b in U_4 independently and take into account that the connection 1-form satisfies to the condition, $\Gamma_{ab} = -\Gamma_{ba}$. It is useful to use the formula (A.1) of Apendix. The variation with respect to θ^a gives the field equation,

$$
\begin{aligned}
&f_0 \mathcal{R}^{bc} \wedge \eta_{abc} + \tau_1(\mathcal{R}^n_b \wedge *(\mathcal{R}^b_n \wedge \theta_a) + *(*\mathcal{R}^b_n \wedge \theta_a) \wedge *\mathcal{R}^n_b) \\
&+\tau_2(2\mathcal{R}_{an} \wedge *(\mathcal{R}^{mn} \wedge \theta_m) - (\mathcal{R}^{rb} \wedge \theta_r) \wedge *(\mathcal{R}^m_b \wedge \theta_m \wedge \theta_a) \\
&- *(*(\mathcal{R}^m_b \wedge \theta_m) \wedge \theta_a) \wedge *(\mathcal{R}^{rb} \wedge \theta_r)) \\
&+\tau_3(2\mathcal{R}^t_n \wedge *(\mathcal{R}^n_a \wedge \theta_t) - (\mathcal{R}^b_l \wedge \theta_m) \wedge *(\mathcal{R}^m_b \wedge \theta^l \wedge \theta_a) \\
&- *(*(\mathcal{R}^m_b \wedge \theta^\lambda) \wedge \theta_a) \wedge *(\mathcal{R}^\beta_l \wedge \theta_m)) \\
&+\tau_4(4\mathcal{R}_{ab} \wedge \theta^b \wedge \theta^b \wedge *(\mathcal{R}^{ln} \wedge \theta_l \wedge \theta_n) + *(*(\mathcal{R}^l_n \wedge \theta_l \wedge \theta^n) \wedge \theta_a) \wedge *(\mathcal{R}^t_b \wedge \theta_t \wedge \theta^b)) \\
&+\tau_5(4\mathcal{R}_{[a|b|} \wedge \theta^l \wedge *(\mathcal{R}^n_{l]} \wedge \theta_n \wedge \theta^b) + *(*(\mathcal{R}^n_l \wedge \theta_n \wedge \theta^b) \wedge \theta_a) \wedge *(\mathcal{R}^t_b \wedge \theta_t \wedge \theta^l)) \\
&+\tau_6(4\mathcal{R}^n_b \wedge \theta^l \wedge *(\mathcal{R}_{al} \wedge \theta_n \wedge \theta^b) + *(*(\mathcal{R}^l_n \wedge \theta^t \wedge \theta^b) \wedge \theta_a) \wedge *(\mathcal{R}_{tb} \wedge \theta_l \wedge \theta^n)) \\
&+\varrho_1(2D * \mathcal{T}_a + \mathcal{T}_n \wedge *(\mathcal{T}^n \wedge \theta_a) + *(*\mathcal{T}^n \wedge \theta_a) \wedge *\mathcal{T}_n) \\
&+\varrho_2(4\mathcal{T}_a \wedge *(\mathcal{T}^b \wedge \theta_b) - 2\theta_a \wedge D * (\mathcal{T}^b \wedge \theta_b) - (\mathcal{T}^l \wedge \theta_l) \wedge *(\mathcal{T}^b \wedge \theta_b \wedge \theta_a)) \\
&+\varrho_2(*(*(\mathcal{T}^b \wedge \theta_b) \wedge \theta_a) \wedge *(\mathcal{T}^l \wedge \theta_l)) \\
&+\varrho_3(2\mathcal{T}^b \wedge *(\mathcal{T}_a \wedge \theta_b) + 2D(\theta^b \wedge *(\mathcal{T}_b \wedge \theta_a)) - (\mathcal{T}_l \wedge \theta_n) \wedge *(\mathcal{T}^n \wedge \theta^l \wedge \theta_a)) \\
&+\varrho_3(*(*(\mathcal{T}^n \wedge \theta^l) \wedge \theta_a) \wedge *(\mathcal{T}_l \wedge \theta_n)) = 0 \,.
\end{aligned}
\tag{2.15}
$$

The second field equation is the result of the variation with respect to Γ^a_b and has the form,

$$
\begin{aligned}
&f_0 D\eta_a^{\ b} + \tau_1 2D * \mathcal{R}^b_a + \tau_2 D(\theta_a \wedge *(\mathcal{R}^{cb} \wedge \theta_c) - \theta^b \wedge *(\mathcal{R}^c_a \wedge \theta_c)) \\
&+\tau_3 D(\theta^c \wedge *(\mathcal{R}^b_c \wedge \theta_a) - \theta^c \wedge *(\mathcal{R}_{ca} \wedge \theta^b)) \\
&+\tau_4 2D(\theta_a \wedge \theta^b \wedge *(\mathcal{R}^{cd} \wedge \theta_c \wedge \theta_d)) \\
&+\tau_5 D(\theta_a \wedge \theta^d \wedge *(\mathcal{R}^c_d \wedge \theta_c \wedge \theta^b) - \theta^b \wedge \theta^d \wedge *(\mathcal{R}^c_d \wedge \theta_c \wedge \theta_a)) \\
&+\tau_6 2D(\theta_c \wedge \theta^d \wedge *(\mathcal{R}^c_d \wedge \theta_a \wedge \theta^b)) \\
&+\varrho_1(\theta^b \wedge *\mathcal{T}_a - \theta_a \wedge *\mathcal{T}^b) + \varrho_2 2\theta^b \wedge \theta_a \wedge *(\mathcal{T}^c \wedge \theta_c) \\
&+\varrho_3(\theta^b \wedge \theta^c \wedge *(\mathcal{T}_c \wedge \theta_a) - \theta_a \wedge \theta^c \wedge *(\mathcal{T}_c \wedge \theta^b)) = 0 \,.
\end{aligned}
\tag{2.16}
$$

3. Plane torsion waves in Riemann–Cartan space-time

As it is shown in [28], [30], it is convinient to choose the special basis to investigate the problem of gravitational waves. This basis is constructed from two null vectors $\bar{e}_0 = \partial_v$, $\bar{e}_1 = \partial_u$ and two space-like vectors $\bar{e}_3 = \partial_x$, $\bar{e}_4 = \partial_y$. The vector $\bar{e}_0$ is covariant constant and has the

direction of the wave ray. Coordinates x and y parametrize the wave surface $(u, v) = \text{const}$. In this basis the metric tensor has the form,

$$g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

In a Riemann space V_4 the plane wave of a metric (see Refs. [1], [30]) is defined as the special case of the metric for the plane-fronted gravitational waves with parallel rays (pp-waves), which in the basis chosen has the form,

$$g = 2H(x, y, u)du^2 + 2dudv - dx^2 - dy^2 . \tag{3.1}$$

Here the coordinate u is considered as the retarded time parameter and can be interpreted as the phase of the wave. The vacuum Einstein equations for the metric (3.1) are equivalent to the equation $H_{xx} + H_{yy} = 0$, where H_{xx}, H_{yy} are the second partial derivatives of H with respect to the corresponding coordinates. The following null coframe corresponds to the metric (3.1),

$$\theta^0 = H\,du + dv , \quad \theta^1 = du , \quad \theta^2 = dx , \quad \theta^3 = dy . \tag{3.2}$$

The Riemann space V_4 with the metric of the plane wave admits the G_5 group of symmetries. This group is generated by the vector fields X with the structure [1],

$$X = (a + b'x + c'y)\partial_v + b\partial_x + c\partial_y , \tag{3.3}$$

where $a = \text{const}$, $b(u)$, $c(u)$ are arbitrary functions and b', c' are their derivatives with respect to u. The G_5-group of symmetries leaves the isotropic hypersurfaces in V_4 to be unchanged. These hypersurfaces describe the wave fronts with the constant amplitude. As the special case of pp-waves, the plane waves have the vanishing expansion, twist and they are shear-free [33].

In Ref. [1] the definition of the plane wave metric is extended to the space with nonvanishing torsion. Let us introduce the following definition [31], [32].

Definition. We shall call a Riemann–Cartan space U_4 as a space U_4 of a plane wave type and its metric and torsion as the plane waves of a metric and torsion, if the metric g_{ab} and the torsion 2-form $\mathcal{T}^a$ of this space admit a five-dimentional group G_5 of symmetries. It means that the following conditions are fullfilled: $\mathrm{L}_X g_{ab} = 0$, $\mathrm{L}_X \mathcal{T}^a = 0$, where L_X denotes the Lie derivative with respect to any vector field X which generates the G_5 symmetry.

The following theorem determines the structure of the plane torsion waves.

Theorem 3.1. The torsion 2-form of U_4 of the plane wave type has the following structure: its trace and pseudotrace vanish and its traceless part depends on two arbitrary functions t and s of the retarded time parameter u and has the form,

$$\overset{(1)}{\mathcal{T}}{}^0 = t(u)\theta^1 \wedge \theta^2 + s(u)\theta^1 \wedge \theta^3 . \tag{3.4}$$

Proof. Let us substitute the vector field (3.3) into the equation $\mathrm{L}_X \mathcal{T}^a = 0$. As a result one obtains the system of the equations. Since b, c, b' and c' can take arbitrary values, these equations imply that all the components of torsion vanish except $T^0{}_{12}$ and $T^0{}_{13}$, which are the functions of u. The straightforward verification shows that the trace and pseudotrace of

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torsion (2.9) of plane waves vanish and only the traceless part of torsion depends on these nonvanishing components, and the dependence has the form (3.4).

The following theorem specifies the conditions under which the plane torsion waves exist.

Theorem 3.2. The metric and torsion of U_4 of plane wave type are the solutions of the field equations of the gravitational theory with the Lagrangian (2.11) with $f_0 \neq 0$ if and only if   (a) the metric of U_4 satisfies the equation $\overset{R}{R}_{ab} = 0$, where $\overset{R}{R}_{ab}$ is the Ricci tensor of the Riemann connection;   (b) the following restriction on the parameters of the Lagrangian (2.11) is valid, $f_0 + \chi_1 = 0$. If the condition (b) is not satisfied then torsion of the plane wave vanishes.

Proof. The connection 1-form of a Riemann–Cartan space U_4 can be decomposed into a sum of the Riemann (Levi-Chivita) connection 1-form $\overset{R}{\Gamma}{}^a_b$ and the contorsion 1-form $\mathcal{K}^a_b$. The latter lineary depends on torsion [26],

$$\Gamma^a_b = \overset{R}{\Gamma}{}^a_b + \mathcal{K}^a_b \,, \quad \mathcal{T}^a =: \mathcal{K}^a_b \wedge \theta^b \,, \quad \mathcal{K}_{ab} = 2\bar{e}_{[a]}\mathcal{T}_{b]} - \frac{1}{2}\bar{e}_a\rfloor\bar{e}_b\rfloor(\mathcal{T}_c \wedge \theta^c) \,. \tag{3.5}$$

In consequence of (3.5) the decomposition of the curvature 2-form in U_4 into the Riemannian and post-Riemannian parts reads,

$$\mathcal{R}^a_b = \overset{R}{\mathcal{R}}{}^a_b + \overset{R}{D}\,\mathcal{K}^a_b + \mathcal{K}^a_c \wedge \mathcal{K}^c_b \,, \tag{3.6}$$

where $\overset{R}{D}$ is the external covariant derivative with respect to the Riemann connection 1-form, and $\overset{R}{\mathcal{R}}{}^\alpha_\beta$ is the Riemann curvature 2-form. By the explicit calculation using (3.2), (3.4) and (3.5) one can verify that all nonvanishing components of the Riemann connection 1-form and the contorsion 1-form are proportional to the basis form $\theta^1 = du$,

$$\overset{R}{\Gamma}{}^0_2 = \overset{R}{\Gamma}{}^2_1 = H_x\theta^1 \,, \qquad \overset{R}{\Gamma}{}^0_3 = \overset{R}{\Gamma}{}^3_1 = H_y\theta^1 \,, \tag{3.7}$$

$$\mathcal{K}^0_2 = \mathcal{K}^2_1 = t\theta^1 \,, \qquad \mathcal{K}^0_3 = \mathcal{K}^3_1 = s\theta^1 \,, \tag{3.8}$$

where H_x and H_y are the first partial derivatives of H with respect to the correspoinding coordinates. In consequence of (3.7) the external covariant derivative with respect to the Riemann connection is also proportional to the basis 1-form $\theta^1 = du$. Therefore all the non-Riemannian terms in (3.6) vanish and the curvature 2-form of U_4 of the plane wave type coinsides with the Riemann curvature 2-form. The only nonvanishing in the basis (3.2) components of the curvature 2-form read,

$$\mathcal{R}^0_2 = \mathcal{R}^2_1 = H_{xx}\theta^2 \wedge \theta^1 + H_{xy}\theta^3 \wedge \theta^1 \,, \quad \mathcal{R}^0_3 = \mathcal{R}^3_1 = H_{xy}\theta^2 \wedge \theta^1 + H_{yy}\theta^3 \wedge \theta^1 \,. \tag{3.9}$$

Therefore in case of the plane waves of a metric and torsion the curvature 2-form $\mathcal{R}^a_b$ in the field equations (2.15) and (2.16) coinsides with the curvature of the Riemann connection $\overset{R}{\mathcal{R}}{}^a_b$. Then in consequence of the identity for the Riemann curvature 2-form

$$\overset{R}{\mathcal{R}}{}^a_b \wedge \theta^b = 0 \,, \tag{3.10}$$

all the terms in the field equation (2.15) with the coefficients τ_2, τ_4 and τ_5 vanish. Moreover, as the result of the curvature 2-form structure (3.9) the terms with the coefficients τ_1, τ_3 and

τ_6 also vanish from the equation (2.15). Taking into account that trace and pseudotrace of torsion vanish in the case considered and that for the tracefree part the condition (2.10) is valid, one can varify that the field equation (2.15) is equivalent to the equation,

$$f_0 \overset{R}{\mathcal{R}}{}^{bc} \wedge \eta_{abc} + 2\,(\varrho_1 + \varrho_3)\,D * \mathcal{T}_a = 0 \ . \tag{3.11}$$

From the structure of the torsion 2-form (3.4) and the connection of the plane wave (3.7), (3.8) it follows that the last term in (3.11) vanishes. Then the equation (3.11) takes the form,

$$f_0 \overset{R}{R}{}_{ab} = 0 \ . \tag{3.12}$$

This proves the part (a) of Theorem 3.2.

Let us now consider the field equation (2.16). In this equation as well as in (2.15) the curvature 2-form coinsides with the curvature of the Riemann connection in V_4. Then due to (3.10) the terms with the coefficients τ_2, τ_4 and τ_5 vanish. Next, using the component representation of the curvature 2-form, together with the identities (2.2)–(2.5), one can show that the terms with the coefficients τ_3 and τ_6 are equal to

$$\tau_3 D(2 * \mathcal{R}_a{}^b + R_{ac}\eta^{bc} - R^{bc}\eta_{ac}) \ , \tag{3.13}$$

$$8\tau_6 D(*\mathcal{R}_a{}^b + R_{ac}\eta^{bc} - R^{bc}\eta_{ac} + \frac{1}{4}R\eta_a{}^b) \ , \tag{3.14}$$

where the notations (2.14) are used. It is necessary to take into account that because of the equation (3.12) the Ricci tensor and the scalar curvature vanish in (3.13) and (3.14). The term with the coefficient f_0 in (2.16) can be transformed with the help of (2.6). Now let us take into account the fact that the trace and pseudotrace of the torsion vanish and also use the following identities for the traceless part of torsion,

$$* \overset{(1)}{\mathcal{T}}{}^b = \theta^c \wedge *(\overset{(1)}{\mathcal{T}}{}_c \wedge \theta^b) \ , \quad \overset{(1)}{\mathcal{T}}{}^c \wedge \eta^b{}_{ac} = \theta_a \wedge * \overset{(1)}{\mathcal{T}}{}^b - \theta^b \wedge * \overset{(1)}{\mathcal{T}}{}_a \ . \tag{3.15}$$

The identities (3.15) can be verified by choosing the component representation and using (2.2)–(2.5). Finally, in the equation obtained let us decompose the connection 1-form into its Riemannian and post-Riemannian parts according to (3.5). As a result the equation (2.16) for the Riemann–Cartan space of the plane wave type becomes,

$$\tau \overset{R}{D} * \overset{R}{\mathcal{R}}{}^b{}_a + 2\tau \left((\bar{e}^{[b} \rfloor \overset{(1)}{\mathcal{T}}{}_{c]}) \wedge * \overset{R}{\mathcal{R}}{}^c{}_a - (\bar{e}^{[c} \rfloor \overset{(1)}{\mathcal{T}}{}_{a]}) \wedge * \overset{R}{\mathcal{R}}{}^b{}_c \right)$$

$$+ (f_0 + \varrho_1 + \varrho_3) \left(* \overset{(1)}{\mathcal{T}}{}_a \wedge \theta^b - * \overset{(1)}{\mathcal{T}}{}^b \wedge \theta_a \right) = 0 \ , \tag{3.16}$$

where $\tau = 2(\tau_1 - \tau_3 - 4\tau_6)$. Let us consider each term of this equation separately. The first term vanishes by virtue of the Bianchi identity in V_4 and the equation (3.12). The second term vanishes due to the structure of the curvature 2-form (3.9) and the torsion 2-form (3.4). As a result the equation (3.16) reduces to the system of the equations,

$$(f_0 + \varrho_1 + \varrho_3)t(u) = 0 \ , \qquad (f_0 + \varrho_1 + \varrho_3)s(u) = 0 \ . \tag{3.17}$$

The rest of the statements of Theorem 3.2 follow from (3.17) due to (2.12) and (3.4).

Corollary. Only the traceless part of torsion can propagate in the form of a plane wave.

Traceless torsion plane waves have the massless quanta.

Proof. Using (2.8) one can decompose the linear term of the Lagrangian (2.7) into the Riemannian and post-Riemannian parts as follows,

$$
\mathcal{R}^a{}_b \wedge \eta_a{}^b = \overset{R}{\mathcal{R}}{}^a{}_b \wedge \eta_a{}^b - 2 \overset{R}{D} \left(* \overset{(2)}{\mathcal{T}}{}^a \wedge \theta_a \right)
$$

$$
+ \overset{(1)}{\mathcal{T}}{}^a \wedge * \overset{(1)}{\mathcal{T}}_a - 2 \overset{(2)}{\mathcal{T}}{}^a \wedge * \overset{(2)}{\mathcal{T}}_a - \frac{1}{2} \overset{(3)}{\mathcal{T}}{}^a \wedge * \overset{(3)}{\mathcal{T}}_a .
\tag{3.18}
$$

Using (3.18) one obtains that the coefficient at the quadratic traceless part of torsion in (2.7) is $f_0 + \chi_1$ and is equal to zero in consequence of Theorem 3.2. This can be interpreted as the fact that the quanta of the corresponding part of torsion are massless.

4. Discussion

In this article the gravitational field equations for the general 10-parametric quadratic Lagrangian are obtained in the first order variational formalism in U_4 in terms of the exterior differential forms. The variational procedure is based on the formula for the commutator of variation of the arbitrary p-form and the Hodge operation, which is proved as the Lemma in the mathematical Appendix. The field equations obtained are used for the analysis of the problem of the plane torsion waves in the quadratic gravitational theories.

For this purpose the definition of the plane waves of a metric and torsion as the Riemann–Cartan space of the plane wave type is given. The structure of the torsion 2-form in the case of the plane waves is determined by Theorem 3.1. From this Theorem it follows that only the traceless part of torsion can propagate in the form of the plane wave. In connection with this result the important question of the possible sources of such waves arises, since the usual types of matter (spinor and electromagnetic fields) can not be the source of the torsion traceless part.

The necessary and sufficient conditions for the existense of the plane torsion waves are found. They are determined by Theorem 3.2. These conditions consist of two parts. The first one requiers the metric of the plane wave to be the plane wave metric with respect to the connection of the Riemann space V_4, whose metric coinsides with the metric of the Riemann–Cartan space U_4. The second condition requiers the constraint to be imposed on the parameters of the quadratic Lagrangian, $f_0 + \chi_1 = 0$. As follows from Corollary of Theorem 3.2 the last condition leads to the fact that the traceless part of torsion must have the massless quanta.

The restriction $f_0 + \chi_1 = 0$ on the Lagrangian parameters plays the important role in analysis of various quadratic gravitational theories with torsion. Thus, in [20] it is shown that if this condition is fulfilled, the spherically symmetric solution for the torsion has the $1/r^2$ form. This leads to the theory without tachyons on the quantum level. In [15] five types of the Lagrangian which lead to the theory without ghosts and tachyons are found. Among them there are two Lagrangians for which this restriction on the constants holds. This leads in turn to the possible propagation of the plane torsion waves. The condition $f_0 + \chi_1 = 0$ is also important in the investigation of particle spectrum in the Poincaré gauge gravitational theory. In [13] it is shown that if this condition is valid then the normal multiplet of torsion can be constructed only from the following combinations of the irreducible components: $(2^-, 1^-, 0^+)$, $(2^-, 1^+, 0^+)$, $(2^-, 0^+, 0^-)$.

Appendix

We consider the arbitrary (in general anholonimic) basis $\bar{e}_\alpha$. For this basis the cobasis of 1-forms θ^β does not consist of total differentials. The metric takes the form $g = g_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$, where $g_{\alpha\beta} = g(\bar{e}_\alpha, \bar{e}_\beta)$. The following Lemma is valid.

Lemma. Let Φ and Ψ be arbitrary p-forms defined on n-dimentional manifold. Then the following variational identity for the commutator of the variation operator and the Hodge star operator is valid,

$$\Phi \wedge \delta * \Psi = \delta\Psi \wedge *\Phi + \delta g_{\sigma\rho}\left(\frac{1}{2}g^{\sigma\rho}\Phi \wedge *\Psi + (-1)^{p(n-p)+\mathrm{Ind}(g)+1}\theta^\sigma \wedge *(*\Psi \wedge \theta^\rho) \wedge *\Phi\right)$$

$$+\delta\theta^\alpha \wedge \left((-1)^p\Phi \wedge *(\Psi \wedge \theta_\alpha) + (-1)^{p(n-p)+\mathrm{Ind}(g)+1}*(*\Psi \wedge \theta_\alpha) \wedge *\Phi\right) . \tag{A.1}$$

Here δ is the operator of the variational derivative and $\mathrm{Ind}(g)$ is the index of the metric g, which is equal to the number of negative eigenvalues of the diagonalized metric.

Proof. Let Ψ be an arbitrary p-form with the component representation

$$\Psi = \frac{1}{p!}\Psi_{\gamma_1\gamma_2...\gamma_p}\theta^{\gamma_1}\theta^{\gamma_2}...\theta^{\gamma_p} . \tag{A.2}$$

Then the Hodge star operator applied to this p-form reads,

$$*\Psi = \frac{\sqrt{|\det g_{\sigma\rho}|}}{(n-p)!p!}g^{\alpha_1\gamma_1}...g^{\alpha_p\gamma_p}\epsilon_{\alpha_1...\alpha_p\beta_1...\beta_{n-p}}\Psi_{\gamma_1...\gamma_p}\theta^{\beta_1} \wedge ... \wedge \theta^{\beta_{n-p}} , \tag{A.3}$$

where $\epsilon_{\alpha_1...\alpha_p\beta_1...\beta_{n-p}}$ are the components of the totally antysymmetric Levi–Chivita density n-form. Let us calculate the variation $\delta\Psi$ using (A.2). Then applying (A.3) one gets,

$$*\delta\Psi = \frac{1}{p!}\eta^{\gamma_1...\gamma_p}\delta\Psi_{\gamma_1...\gamma_p} + \frac{1}{(p-1)!}\Psi_{\gamma_1...\gamma_p}\left(\bar{e}_\lambda\rfloor\delta\theta^{\gamma_1}\right)\eta^{\lambda\gamma_2...\gamma_p} . \tag{A.4}$$

The calculation of the variation of (A.3) is more complicated since the components of the metric tensor must be also varied. After rather tedious calculations one gets,

$$\delta * \Psi = \delta g_{\sigma\rho}\left(\frac{1}{2}(*\Psi)g^{\sigma\rho} - \frac{1}{(p-1)!}\Psi^{\sigma\gamma_1...\gamma_{p-1}}\eta^\rho{}_{\gamma_1...\gamma_{p-1}}\right)$$

$$+\frac{1}{p!}\eta^{\gamma_1...\gamma_p}\delta\Psi_{\gamma_1...\gamma_p} + \delta\theta^\beta \wedge \left(\frac{1}{p!}\Psi^{\gamma_1...\gamma_p}\eta_{\gamma_1...\gamma_p\beta}\right) . \tag{A.5}$$

Here the p-form $\eta^{\gamma_1...\gamma_p} = *(\theta^{\gamma_1} \wedge ... \wedge \theta^{\gamma_p})$ is introduced. Let us multiply externally from the left (A.4) and (A.5) by the p-form Φ and take into account that $\Phi \wedge *\delta\Psi = \delta\Psi \wedge *\Phi$. Comparing the expressions obtained and using the well-known relations,

$$\bar{e}_\alpha\rfloor * \Psi = *(\Psi \wedge \theta_\alpha) , \qquad * * \Psi = (-1)^{p(n-p)+\mathrm{Ind}(g)}\Psi , \tag{A.6}$$

one gets the formula (A.1) of Lemma.

References

[1] Adamovich W 1980 *Gen. Relat. Gravit.* **12** 677

[2] Chen M, Chern D, Hsu R and Yeung W B 1983 *Phys. Rev.* D **28** 2094

[3] Singh P and Griffiths J P 1990 *Gen. Relat. Gravit.* **22** 269, 947

[4] Zhytnikov V V 1994 *J. Math. Phys.* **35** 6001

[5] Babourova O V, Klimova E A, Frolov B N 1997 *Nauchnye trudy MPGU. Ser. Est. nauki,* (Moscow: Prometey) p 142 [in Russian]

[6] Frolov B N 1963 *Vestnik Mosk. Univ. Ser. fizika, astron.* No 6, 48 [in Russian]

[7] Frolov B N 1967 *Sovremennye problemy gravitatsii.* Proc. of the 2nd Soviet Grav. Conf. (Tbilisi: Izd. Tbil. Univ.) p. 270 [in Russian]

[8] Hayashi K 1968 *Progr. Theor. Phys.* **39** 495

[9] Yang C N 1974 *Phys. Rev. Lett.* **33** 445

[10] Fairchild E E 1976 *Phys. Rev.* D **14** 384

[11] von der Heyde P 1976 *Z. Naturforsch.* **31a** 1725

[12] Frolov B N 1977 *Izvestija vyssh. uch. zaved. (Fizika)* No 3, 154, 155; No 5, 148 [in Russian]; 1978 *Acta Phys. Polon.* **B9** 823

[13] Hayashi K and Shirafuji T 1980 *Progr. Theor. Phys.* **64** 866, 883, 1435, 2222; 1981 **65** 525, **66** 318, 2258

[14] Minkevich A V 1980 *Phys. Lett.* **80A** 232

[15] Sezgin E and van Nieuwenhuizen P 1980 *Phys. Rev.* D **21** 3269

[16] Rauch R and Nieh H T 1981 Phys. Rev. D **24** 2029

[17] Tseytlin A A 1982 *Phys. Rev.* D **26** 3327

[18] Garecki J 1983 *Acta Phys. Polon.* B **13** 397; B **14** 713; 1985 **B16** 699

[19] Yan M L 1983 *Commun. Theor. Phys.* (Beijing, China) **2** 1281

[20] Zhang Y Z 1983 *Phys.Rev.* D **28** 1866

[21] Szczyrba V 1987 *Phys. Rev.* D **36** 351

[22] Obukhov Yu N, Ponomariov V N and Zhytnikov V V 1989 *Gen. Relat. Gravit.* **21** 1107

[23] Basombrio F G 1980 *Gen. Relat. Gravit.* **12** 109

[24] Ivanenko D and Sardanashvily G 1983 *Phys. Rep.* **94** 1

[25] Ivanenko D D, Pronin P I and Sardanashvily G A *Kalibrovochnaya teoriya gravitatsii* (Moscow: Izd. Mosk. Univ. 1985)

[26] Hehl F W, McCrea J L, Mielke E W and Ne´eman Yu 1995 *Phys. Reports* **258** 1

[27] Trautman A 1973 *Differential Geometry. Symposia Mathematica* **12** (London: Academic Press) p 139

[28] Ehlers J and Kundt W 1962 Exact solutions of the gravitational field equations *Gravitation: an introduction to current research* ed L Witten (New York, London: John Wiley)

[29] McCrea J D 1992 *Class. Quantum Grav.* **9** 553

[30] Kramer D, Stephani H, MacCallum M and Herlt E *Exact Solutions of the Einsteins Field Equations* ed E Schmutzer (Berlin: Deutscher Verlag der Wissenschaften 1980)

[31] Babourova O V, Frolov B N and Klimova E A 1997 *GR15 Abst. of Plenary Lectures and Contr. Papers, 1997, Pune* p 103

[32] Babourova O V, Klimova E A, Frolov B N 1998 *Izvestija vyssh. uch. zaved. (Fizika)* to be published [in Russian]

[33] Zakharov V D *Gravitatsionnye volny v teorii tyagoteniya Einshteina* (Moscow: Nauka 1972) [in Russian]